Riemannian Manifolds and Affine Connections

Ashwin Devaraj

Advised by Dan Weser

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1 Introduction

1.1 Overview of Topics

1.2 Prerequisites

These notes assume a good knowledge of multivariable calculus, basic linear algebra, and point-set topology. To fully appreciate the motivations behind this material, it would be helpful to understand basic differential geometry in Euclidean space (i.e. regular curves and surfaces, curvature, tangent spaces). The notation and definitions we use in these notes follow Do Carmo's *Riemannian Geometry*. As per this text, the term *differentiable* is taken to mean *smooth*.

2 Differentiable Manifolds

2.1 Basic Definitions

The first step in our generalization of differential geometry in Euclidean space is to take an arbitrary set that locally "looks" like Euclidean space with a differentiable structure. This motivates our definition of a *differentiable manifold*, which can be viewed as a generalization of the regular surface in $\mathbb{R}^3$.

Definition 2.1. A *differemtiable manifold* of dimension n is a set M and a family of injective mappings $\mathbf{x}_\alpha : U_\alpha \to M$ where each U_α is an open subset of $\mathbb{R}^n$ and the following properties hold:

- $\bigcup_\alpha \mathbf{x}_\alpha(U_\alpha) = M$. This means that the images of the open sets U_α cover the set M.
- For any pair α and β where intersection $W := \mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta)$ is nonempty, the sets $\mathbf{x}_\alpha^{-1}(W)$ and $\mathbf{x}_\beta^{-1}(W)$ are both open and the function $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha : \mathbf{x}_\alpha^{-1}(W) \to \mathbf{x}_\beta^{-1}(W)$ is differentiable. This is illustrated in Figure 1.
- The family $\{(\mathbf{x}_\alpha, U_\alpha)\}$ is maximal.

The maximal family $\{(x_\alpha, U_\alpha)\}$ associated with M is called a *differentiable structure*. This differentiable structure induces a topology on M, namely that a set S in M is open if $x_\alpha^{-1}(S)$ is open in $\mathbb{R}^n$ for all α.

An n-dimensional manifold M may be referred to by the name M^n as short-hand. Since we do not deal with product manifolds in these notes, this notation does not introduce any ambiguity.

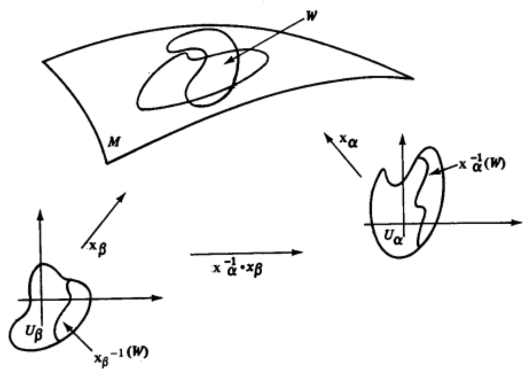

Figure 1: Differentiability Condition

Given a manifold M with differentiable structure, it is natural for us to define the notion of a differentiable function $f : M_1^n \to M_2^m$. Intuitively, a function f is differentiable if rewriting it in terms of any parametrizations of M_1 and M_2 in open subsets $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}^m$ produces a differentiable function from U_1 to U_2.

Definition 2.2. Let M_1^n and M_2^m be differentiable manifolds. Then a function $f : M_1^n \to M_2^m$ is *differentiable* at $p \in M_1$ if for any parametrization $\mathbf{y} : V \subset \mathbb{R}^m \to M_2$ at $f(p)$, there exists a parametrization $\mathbf{x} : U \subset \mathbb{R}^n \to M_1$ at p such that $f(\mathbf{x}(U)) \subset \mathbf{y}(V)$ such that the function $\mathbf{y}^{-1} \circ f \circ \mathbf{x} : U \to \mathbb{R}^m$ is differentiable at $\mathbf{x}^{-1}(p)$. Figure 2 illustrates this definition.

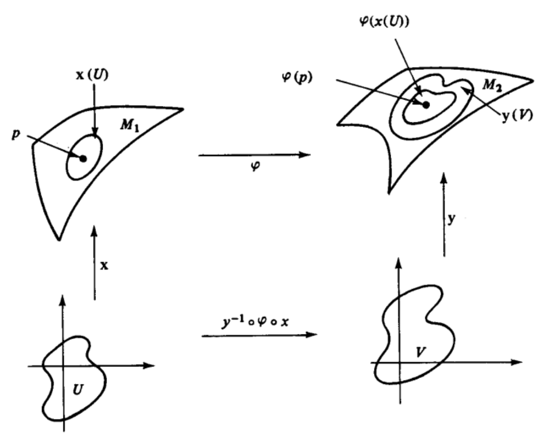

Figure 2: Showing Differentiability of Function between Manifolds

The next properties of regular surfaces that we seek to generalize are the notions of a tangent vector and tangent space at a point on a manifold M. In Euclidean space, the tangent vector of a point on a regular surface is defined to be the derivative of a curve in the surface passing through the point of interest. Tangent vectors are thus vectors in the ambient space $\mathbb{R}^3$ and the tangent plane is a 2-dimensional subspace.

Arbitrary differentiable manifolds do not necessarily have an ambient space in which tangent vectors can be defined, so we must come up with another way to define them. Note that for any $v = (v_1, .., v_n) \in \mathbb{R}^n$, we can define a map that takes a real-valued function on $\mathbb{R}^n$ and outputs a real value, namely the directional derivative in the direction v. More specifically, we take a function $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that $\alpha'(0) = v$ and get

$$vf = \frac{d(f \circ \alpha)}{dt}\Big|_{t=0} = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}\Big|_{t=0}$$

Thus each v is associated with a unique map taking functions to their directional derivatives in the direction of v. This suggests that we can define tangent vectors on a manifold as a function on differentiable functions on M, a formulation that does not depend on M being embedded in an ambient space.

Definition 2.3. Let M be a differentiable manifold, $p \in M$, and $\alpha : (-\varepsilon, \varepsilon) \to M$ be a curve such that $\alpha(0) = p$. Let $\mathcal{D}$ denote the set of real-valued functions that are differentiable at p. Then the following function $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ is a *tangent vector* at the point p:

$$\alpha'(0)f = \frac{d(f \circ \alpha)}{dt}\Big|_{t=0}$$

The set of such functions T_pM is called the *tangent plane* at point p. It is an n-dimensional vector space.

It turns out that if we pick a local parametrization $\mathbf{x}$ in a neighborhood of the point p, there exists a natural basis for the tangent space. Let $f \in \mathcal{D}$, p a point in M, and $\alpha : (-\varepsilon, \varepsilon) \to M$ such that $\alpha(0) = p$. Furthermore, let $q = \mathbf{x}^{-1}(p) \in \mathbb{R}^n$. Then in terms of the parametrization $\mathbf{x}$, we write $f \circ \mathbf{x} = f(x_1, ..., x_n)$ and $\mathbf{x}^{-1} \circ \alpha(t) = (x_1(t), ..., x_n(t))$. Then $f \circ \alpha(t) = (f \circ \mathbf{x}) \circ (\mathbf{x}^{-1} \circ \alpha)(t) = f(x_1(t), ..., x_n(t))$. We thus compute the function $\alpha'(0)$ to be

$$\begin{aligned}
\alpha'(0)f &= \frac{d(f \circ \alpha)}{dt}\Big|_{t=0} \\
&= \frac{d}{dt} f(x_1(t), ..., x_n(t))\Big|_{t=0} \\
&= \sum_{i=1}^{n} x_i'(0) \frac{\partial f}{\partial x_i} \\
&= \left(\sum_{i=1}^{n} x_i'(0) \left(\frac{\partial}{\partial x_i} \right) \right) f,
\end{aligned}$$

where each $\frac{\partial}{\partial x_i}$ is the tangent vector at p of the coordinate curve $t \mapsto x(0, ..., t, ...0)$. Thus $\left\{ \frac{\partial}{\partial x_i} \right\}_{i=1}^{n}$ is a basis for T_pM.

Now that we have a notion of tangent vectors and tangent spaces, we can define the differential of a differentiable function from M_1^n to M_2^m. As in the case of regular surfaces in $\mathbb{R}^3$, the differential at a point is a linear map between tangent spaces.

Definition 2.4. Let M_1^n and M_2^m be differentiable manifolds and f a differentiable function from M_1 to M_2. Let $p \in M_1$ and $\alpha : (-\varepsilon, \varepsilon) \to M_1$ such that $\alpha(0) = p$. Let $v = \alpha'(0)$ and $\beta(t) = f \circ \alpha(t)$. We define the *differential* of f at point p as $df_p(v) = \beta'(0)$. df_p is a linear map from T_pM_1 to $T_{f(p)}M_2$ whose definition only depends on the input vector v and not the specific curve α.

2.2 Vector Fields

Every differentiable n-manifold M has associated with it another differentiable $2n$-manifold TM that is constructed by "gluing" the tangent plane to every point of M. Within the scope of these notes, this new manifold is useful to us only in the context of defining vector fields, but it has a wealth of nice properties, such as being orientable even if M is not.

Definition 2.5. Let M denote a differentiable n-manifold. We define the *tangent bundle* TM to be the set $\{(p, v) : p \in M, v \in T_pM\}$. The tangent bundle is a $2n$-dimensional differentiable manifold with a differentiable structure $\{(\mathbf{y}_\alpha, U_\alpha \times \mathbb{R}^n)\}$ where

$$\mathbf{y}_\alpha(x_1, ...x_1, u_1, ..., u_n) = \left(\mathbf{x}_\alpha(x_1, ..., x_n), \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}\right)$$

with $\left\{\frac{\partial}{\partial x_i}\right\}$ describing the basis of each T_pM with respect to parametrization $\mathbf{x}_\alpha$.

Definition 2.6. A *vector field* X on differentiable manifold M is a function that assigns a tangent vector from T_pM to each point $p \in M$. It can be described as a function from M to TM. X is differentiable if the mapping $X : M \to TM$ is.

There are 2 kinds of operations that can be done with a vector field X and differentiable function $f : M \to \mathbb{R}$. One is scalar multiplication of X by the value of f at each point. This is denoted as $fX : M \to TM$. A second kind of operation that can be done is to create a new function $Xf : M \to \mathbb{R}$ in which f is mapped to its directional derivative in the direction of X at each point. This is denoted as Xf.

Thus a vector field on M can be viewed as an operator on the space of differentiable real-valued functions on M, but the composition of these operators does not necessarily produce another field. There is a way, however, to combine 2 vector fields to provide another.

Theorem 2.7. *Let M be a differentiable manifold and X, Y be differentiable vector fields. Then $XY - YX$ is a differentiable vector field called the Lie bracket denoted by $[X, Y]$. The Lie bracket satisfies the following properties:*

- $[X, Y] = -[Y, X]$
- $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- $[fX + gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$

Definition 2.8. Let M^n be a differentiable manifold and $c : I \to M$ be a differentiable curve on M, where I is some interval in $\mathbb{R}$. A *vector field* on c is a function that assigns to each point $c(t)$ an element of $T_{c(t)}M$. This vector field is *differentiable* if for every differential function f on M, the map $t \mapsto V(t)f$ is differentiable over I. The vector field $dc(d/dt)$ is called the *velocity field* of c.

3 Adding Geometric Structure

3.1 Motivation

Now that we have developed a way to describe abstract spaces with local Euclidean structure, we are now ready to define new structures with which geometry can be done. The basic tools needed to do geometry are rulers and protractors, that is, tools used to compute length and angle. In Euclidean space, the inner product (or dot product) provides a way to do both. More specifically, for any two vectors $u, v \in \mathbb{R}^n$:

$$\cos(\theta) = \frac{\langle u, v\rangle}{\|u\|\|v\|}, \quad d(u, v) = \|u - v\|$$

where $\|u\| = \langle u, u\rangle^{1/2}$. Naturally, our next step is to introduce the notion of an inner product to differentiable manifolds.

3.2 The Riemannian Metric

Definition 3.1. Let M^n be a differentiable manifold. A *Riemannian metric* is an association of an inner product $\langle\ ,\ \rangle_p$ to each point $p \in M$ that varies differentiably. To illustrate what this means, we pick some $p \in M$ and a local parametrization $\mathbf{x} : U \to M$ containing p. Then for all $i, j \in \{1, ..., n\}$, $\frac{\partial}{\partial x_i}(\mathbf{x}(q)) = d\mathbf{x}_q(0, ...1, ...0) \in T_{\mathbf{x}(q)}M_1$ and $\langle \frac{\partial}{\partial x_i}(\mathbf{x}(q)), \frac{\partial}{\partial x_i}(\mathbf{x}(q))\rangle_{\mathbf{x}(q)}$ is a differentiable function from U to $\mathbb{R}$. A differentiable manifold with a Riemannian metric is called a *Riemannian manifold.*

For a specific choice of parametrization $\mathbf{x} : U \subset \mathbb{R}^n \to M$ in the neighborhood of some $p \in M$, we can express the inner product $\langle \ , \ \rangle_p$ in this neighborhood as a matrix. Let a_{ij} denote $\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_p$. Then for any $v, w \in T_pM$, $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x_i}$ and $w = \sum_{j=1}^n w^j \frac{\partial}{\partial x_j}$, where $v^i, w^j \in \mathbb{R}$. Then

$$\begin{aligned}
\langle v, w \rangle_p &= \langle \sum_{i=1}^n v^i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w^j \frac{\partial}{\partial x_j} \rangle_p \\
&= \sum_{i=1}^n v^i \sum_{j=1}^n w^j \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_p \\
&= (v')^T A w'
\end{aligned}$$

where v', w' are column vectors of coefficients v^i, w^j and A is a matrix such that $A_{ij} = a_{ij}$. Note that since the inner product is symmetric and positive definite, so is the matrix A.

The simplest example of a Riemannian manifold is $\mathbb{R}^n$ with the standard inner product. At every point $p \in \mathbb{R}^n$, for basis vectors e_i and e_j of $T_p\mathbb{R}^n = \mathbb{R}^n$, $\langle e_i, e_j \rangle = \delta_i^j$, where δ_i^j denotes the Kronecker delta function, which is 1 if $i = j$ and 0 otherwise. In matrix form, this metric corresponds to the identity matrix. The metric is constant over the entire space and thus trivially differentiable.

4 Connections

4.1 A Brief Interlude

One common phenomenon in mathematics is that in an effort to generalize concepts, what is a theorem in the less abstract case then becomes part of the definition in the generalized case. Take for example the notion of angles between vectors. In $\mathbb{R}^2$ and $\mathbb{R}^3$, the formula for the angle between 2 vectors u, v is given by

$$\cos(\theta) = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

This is a theorem that can be proved using basic Euclidean geometry and the Law of Cosines. In other words, we had a pre-existing physical notion of what an angle is and had to prove that the angle between vectors could be computed by the above formula. In an abstract inner product space, however, we do not have a physical notion of what an angle. Thus we define the notion of an angle between vectors to be that which is computed by the formula that was proven in $\mathbb{R}^2$ and $\mathbb{R}^3$, replacing the inner product in $\mathbb{R}^n$ with that provided by the inner product space.

A similar technique is used to define a family directional derivative operators on vector fields on differentiable manifolds. These are called affine connections. It turns out that when we impose additional constraints to connections on Riemannian manifolds, we obtain a unique affine connection.

4.2 Affine Connections

Before we define affine connections, we first discuss the notion of directional derivatives in Euclidean space and generalize this definition to directional derivatives on vector fields.

Definition 4.1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. Then the *directional derivative* in the direction of vector v is

$$\nabla_v f(x) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

$\nabla_v f$ is thus also a function from $\mathbb{R}^n$ to $\mathbb{R}^m$.

This definition can be extended to 2 differentiable vector fields X and Y in $\mathbb{R}^n$, where the directional derivative of one vector field at a point is calculated in the direction of the other vector field at that same point.

Definition 4.2. Let X and Y be differentiable vector fields from $\mathbb{R}^n$ to $\mathbb{R}^n$. Then the *directional derivative* of Y in the direction of X at point p is

$$(\nabla_X Y)(p) = \lim_{t \to 0} \frac{Y(p + tX(p)) - Y(p)}{t}$$

$\nabla_X Y$ is thus also a vector field from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Thus ∇ can be viewed as an operator that takes 2 vector fields and produces a new one: $\nabla(X,Y) = \nabla_X Y$. This operator satisfies a variety of familiar properties.

Theorem 4.3. *Let $\mathfrak{X}$ denote the set of differentiable vector fields from $\mathbb{R}^n$ to $\mathbb{R}^n$, X,Y, and Z be elements of $\mathfrak{X}$, and f, g denote differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$. Furthermore, let $\nabla : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ denote the operator discussed earlier. Then the following properties hold:*

- *(Linearity in direction)* $\nabla(fX + gY, Z) = f\nabla(X,Z) + g\nabla(Y,Z)$
- *(Linearity in vector field)* $\nabla(X, Y + Z) = \nabla(X,Y) + \nabla(Y,Z)$
- *(Product rule)* $\nabla(X, fY) = f\nabla(X,Y) + (Xf)Y$

Our limit definition of a directional derivative operator on vector fields contains operations that work in Euclidean space but do not immediately generalize to differentiable manifolds. The term $p + tX(p)$ makes sense in $\mathbb{R}^n$ since $\mathbb{R}^n$ is an affine space, meaning you can add vectors to points to get new points. On a manifold, adding a vector from T_pM to p is not well-defined. $Y(p+tX(p)) - Y(p)$ makes sense in $\mathbb{R}^n$ since the tangent space at every point is identical. On a manifold however, addition of vectors from distinct tangent spaces T_pM and T_qM is not well-defined.

Thus, our strategy is to introduce a directional derivative operator on abstract manifolds by merely making the properties proven in Theorem 4.3 part of the definition.

Definition 4.4. Let M be a differentiable manifold and $\mathfrak{X}(M)$ the set of differentiable vector fields on M. Let $X, Y, Z \in \mathfrak{X}(M)$ and f, g be differentiable real-valued functions on M. $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, which notationally can be expressed as $\nabla(X,Y)$ or $\nabla_X Y$, is an *affine connection* if it satisfies the following properties:

- $\nabla(fX + gY, Z) = f\nabla(X,Z) + g\nabla(Y,Z)$
- $\nabla(X, Y + Z) = \nabla(X,Y) + \nabla(Y,Z)$
- $\nabla(X, fY) = f\nabla(X,Y) + (Xf)Y$

As we will now see, a choice of affine connection provides a unique method to calculate the derivative of a vector field along a curve satisfying the expected properties of differentiation.

Theorem 4.5. *Let M be a differentiable manifold with affine connection ∇, $c : I \to M$ a differentiable curve on M, and V a vector field defined along c. Then there exists a unique vector field $\frac{DV}{dt}$ along c, called the covariant derivative of V along c, that satisfies the following properties:*

- $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$
- $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$
- *If V is the restriction of some vector field Y defined on M to the curve c, i.e. $V(t) = Y(c(t))$, then $\frac{DV}{dt} = \nabla_{dc/dt} Y$.*

We thus have a notion of differentiating vector fields along curves even though the tangent space at every point on the curve is possibly distinct. Thus the affine connection provides a way to differentiate vector fields along a curve "connecting" distinct tangent spaces. With the notion of a covariant derivative, we can discuss the notion of parallel vector fields.

Definition 4.6. Let M be a differentiable manifold with affine connection ∇, and $c : I \to M$ be a differentiable curve. Then a vector field V along c is *parallel* if $\frac{DV}{dt} = 0$ over I.

Figure 3 shows an example of a parallel vector field along a curve in $\mathbb{R}^2$.

4.3 The Riemannian Connection

Though the affine connection is a useful step towards defining directional differentiation on Riemannian manifolds, the fact that there is not a single canonical choice of connection on differentiable manifolds means that we must impose additional constraints that have to do with the Riemannian metric.

Definition 4.7. Let M be a differentiable manifold with connection ∇ and Riemannian metric $\langle\ ,\ \rangle$. ∇ is *compatible with metric* $\langle\ ,\ \rangle$ if for any differentiable curve c and pair of parallel vector fields P, P' along c, $\langle P, P' \rangle$ is constant.

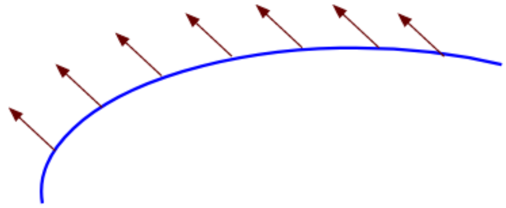

Figure 3: Parallel Vector Field

The intuition behind this condition is that it is equivalent to the statement that the connection agrees with the product rule on the inner product.

Theorem 4.8. *Let M be a differentiable manifold with affine connection ∇ and Riemannian metric $\langle\ ,\ \rangle$. The following statements are equivalent.*

(a) *∇ is compatible with metric $\langle\ ,\ \rangle$.*

(b) *For any differentiable curve c and differentiable vector fields V and W defined along c, $\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle$.*

(c) *For all $X, Y, Z \in \mathfrak{X}(M)$, $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.*

Definition 4.9. An affine connection ∇ on a differentiable manifold M is said to be *symmetric* if for all $X, Y \in \mathfrak{X}(M)$, $\nabla_X Y - \nabla_Y X = [X, Y]$.

We now show that there is a unique choice of connection that is both symmetric and in agreement with the Riemannian metric.

Theorem 4.10 (Fundamental Theorem of Riemannian Geometry)**.** *let M be a Riemannian manifold with Riemannian metric $\langle\ ,\ \rangle$. Then there exists a unique affine connection, called the Levi-Civita connection, that is both symmetric and in agreement with the metric.*

Proof. We first suppose that an affine connection ∇ exists satisfying the 2 conditions. Then for any $X, Y, Z \in \mathfrak{X}(M)$, the following equations hold since ∇ agrees with the metric:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle$$

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

We add the first 2 equations and subtract the last to get:

$$\begin{aligned} X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \\ &= (\langle \nabla_X Z, Y\rangle - \langle \nabla_Z X, Y\rangle) + (\langle \nabla_Y Z, X\rangle - \langle \nabla_Z Y, X\rangle) + \\ &\quad (\langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle) + (\langle \nabla_Y X, Z\rangle + \langle Z, \nabla_Y X\rangle) \\ &= \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle + 2\langle Z, \nabla_Y X\rangle \quad \text{(by symmetry of } \nabla) \end{aligned}$$

This equation can be rewritten as

$$\langle Z, \nabla_Y X\rangle = \frac{1}{2}\left(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle\right)$$

This equation shows that the connection, if it exists, is unique, since the choice of Z is arbitrary. We prove existence by just defining ∇ to satisfy the equation. $\square$

5 Summary

In these notes, we developed the idea of a differentiable manifold to generalize the idea of regular surfaces to abstract manifolds. We then developed the tools to measure lengths and angles on these manifolds with the Riemannian metric. Finally, we explored the properties of directional differentiation in $\mathbb{R}^n$, used these properties to define such an operation (i.e., the affine connection) on differentiable manifolds, and concluded by showing that a unique such operation exists that agrees with the Riemannian metric on Riemannian manifolds. The structures outlined in these notes can then be used to introduce other concepts from the geometry on regular surfaces, such as curvature and geodesics.

6 Figures Cited

- Figure 1: *Riemannian Geometry* by Do Carmo
- Figure 2: *Riemannian Geometry* by Do Carmo
- Figure 3: drawn by Ashwin Devaraj